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FUZZY STABILITY OF C^*- TERNARY ALGEBRA HOMOMORPHISMS

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Abstract

In this paper, we consider the fuzzy stability of C^*- ternary algebra homomorphisms of the following Cauchy-Jensen functional equation:

$$2f\left(\frac{\lambda x + \lambda y}{2} + \lambda z\right) = \lambda f(x) + \lambda f(y) + 2\lambda f(z).$$

1. Introduction

In 1984, Katsaras defined a fuzzy norm on a linear space in [10]. At the same year, Wu and Fang [17] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. Since then, many mathematicians have discussed fuzzy metrics and norms on a linear

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space from various points of view ([7], [11], [18]). In 2003, Bag and Samanta [4] modified the definition of Cheng and Mordeson [6] by removing a regular condition.

The study of stability problems for functional equations are related to a question of Ulam [16] concerning the stability of group of homomorphisms, which was affirmatively answered for Banach spaces by Hyers [9]. Subsequently, the result of Hyers was generalized for unbounded control functions by Aoki [2], and by Rassias [15]. The paper of Rassias [15] has provided a great influence on the development of the very active area of Hyers-Ulam-Rassias stability of functional equations. In 1994, a generalization of Rassias theorem was obtained by Gavruta [8] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ with a general control function $\varphi(x, y)$. Several stability results have been recently obtained for various equations and mappings with more general domains and ranges (see [1], [3], [13]).

In the following, we will give some notations that are needed in this paper. The following notion of a fuzzy norm is taken from [4].

Definition 1.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a *fuzzy norm* on X, if for all $x, y \in X$ and all $s, t \in \mathbb{R}$, we have

(N1) $N(x, c) = 0,$ for $c \leq 0;$

(N2) $x = 0,$ if and only if $N(x, c) = 1$ for all $c > 0;$

(N3) $N(cx, t) = N\left(x, \frac{t}{|c|}\right),$ if $c \neq 0;$

(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\};$

(N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1;$

(N6) For $x \neq 0$, $N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

A fuzzy normed linear space is a pair (X, N), where X is a real linear space and N is a fuzzy norm on X. One may regard $N(x, t)$ as the truth value of the statement of the norm of x is less than or equal to the real number t.

Example 1.2. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, x \in X, \\ 0, & t \leq 0, x \in X, \end{cases}$$

is a fuzzy norm on X.

Example 1.3. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} 0, & t < 0, \\ \dfrac{t}{\|x\|}, & 0 < t \leq \|x\|, \\ 1, & t > \|x\|, \end{cases}$$

is a fuzzy norm on X.

Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be *convergent*, if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1,$ for all $t > 0.$ In this case, x is called the *limit of the sequence* $\{x_n\}$ and we denote it by $N - \lim x_n = x$.

A sequence $\{x_n\}$ in a fuzzy normed space (X, N) is called *Cauchy*, if for each $\epsilon > 0$ and each $t > 0,$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $p > 0,$ we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called a *fuzzy Banach space*.

We also need the following knowledge about C^*-ternary algebras (see [14], [5], [19]). Following the terminology of [1], a non-empty set G with a ternary operation $[\cdot, \cdot, \cdot] : G \times G \times G \to G$ is called a *ternary groupoid* and is denoted by $(G, [\cdot, \cdot, \cdot])$. The ternary groupoid $(G, [\cdot, \cdot, \cdot])$ is called *commutative*, if $[x_1, x_2, x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$, for all $x_1, x_2, x_3 \in G$ and all permutations σ of $\{1, 2, 3\}$.

If a binary operation $\circ$ is defined on G such that $[x, y, z] = (x \circ y) \circ z$ for all $x, y, z \in G$, then we say that $[\cdot, \cdot, \cdot]$ is derived from $\circ$. We say that $(G, [\cdot, \cdot, \cdot])$ is a *ternary semigroup*, if the operation $[\cdot, \cdot, \cdot]$ is *associative*, i.e., if $[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$ holds for all $x, y, z, u, v \in G$ (see [5]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [1], [19]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x, y, z] = \langle x, y \rangle z$. If a C^*-ternary algebra $(A, [,,])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y = [x, e, y]$ and $x^* = [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] = x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary algebra homomorphism*, if $H([x, y, z]) = [H(x), H(y), H(z)]$ for all $x, y, z \in A$.

In this paper, we will establish a fuzzy version of a generalized Hyers-Ulam-Rassias stability for Cauchy-Jensen functional equation

$$2f\left(\frac{\lambda x + \lambda y}{2} + \lambda z\right) = \lambda f(x) + \lambda f(y) + 2\lambda f(z), \tag{1.1}$$

in the fuzzy normed linear space setting. Fuzzy stability of Jensen functional equations has been discussed in [12].

Assume that X be a linear space and (Y, N) be a fuzzy Banach space. Throughout this paper, for a given mapping $f : X \to Y,$ we define

$$C_\lambda f(x, y, z) = 2f\left(\frac{\lambda x + \lambda y}{2} + \lambda z\right) - \lambda f(x) - \lambda f(y) - 2\lambda f(z),$$

for all $\lambda \in \mathbb{T}^1 = \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x, y, z \in X.$

2. Main Results

In this section, we will prove the fuzzy Hyers-Ulam-Rassias stability of C^*-ternary algebra homomorphisms for Cauchy-Jensen functional Equation (1.1).

Theorem 2.1. *Let X be a linear space and (Z, N') be a fuzzy normed space. Let $\varphi : X^3 \to Z$ be a function such that for some $0 < \alpha < 2$*

$$N'(\varphi(2x, 2y, 2z), t) \geq N'(\alpha\varphi(x, y, z), t), \tag{2.1}$$

$$\lim_{n\to\infty} N'\left(\frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z), t\right) = 1, \tag{2.2}$$

for all $x, y, z \in X$ and all $t > 0.$ Suppose (Y, N) is a fuzzy Banach space and $f : X \to Y$ is an odd function such that

$$N(C_\lambda f(x, y, z), t) \geq N'(\varphi(x, y, z), t), \tag{2.3}$$

and

$$N(f([x, y, z]) - [f(x), f(y), f(z)], s) \geq N'(\varphi(x, y, z), s), \tag{2.4}$$

for all $x, y, z \in X$ and all $t, s > 0.$ Then, there is a unique C^*-ternary homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x),\, t) \geq N'\left(\frac{\varphi(x,\, x,\, x)}{2 - \alpha},\, t\right), \tag{2.5}$$

for all $x \in X$ *and* $t > 0$.

Proof. Put $\lambda = 1$ and $x = y = z$ in (2.3), we have

$$N(2f(2x) - 4f(x),\, t) \geq N'(\varphi(x,\, x,\, x),\, t), \tag{2.6}$$

for all $x \in X$ and all $t > 0$. Using (2.1) and induction on n, it is not difficult to show that

$$N'(\varphi(2^n x,\, 2^n x,\, 2^n x),\, t) \geq N'(\alpha^n \varphi(x,\, x,\, x),\, t), \tag{2.7}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^{n-1}x$ in (2.6) and using (2.7), we get

$$N(2f(2^n x) - 4f(2^{n-1}x),\, t) \geq N'(\alpha^{n-1}\varphi(x,\, x,\, x),\, t), \tag{2.8}$$

for all $x \in X$ and all $t > 0$. It follows from (2.8) that

$$N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}},\, \frac{t}{2^{n+1}}\right) \geq N'\left(\frac{1}{\alpha^2}\varphi(x,\, x,\, x),\, \frac{t}{\alpha^{n+1}}\right),$$

and hence,

$$N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^{n-1}x)}{2^{n-1}},\, \left(\frac{\alpha}{2}\right)^{n+1} t\right) \geq N'\left(\frac{1}{\alpha^2}\varphi(x,\, x,\, x),\, t\right),$$

for all $n \geq 1$, $x \in X$, and $t > 0$. Thus

$$N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m},\, \sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k+1} t\right)$$

$$= N\left(\sum_{k=m+1}^{n}\left(\frac{f(2^k x)}{2^k} - \frac{f(2^{k-1}x)}{2^{k-1}}\right),\, \sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k+1} t\right)$$

$$\geq N'\left(\frac{1}{\alpha^2}\varphi(x,\, x,\, x),\, t\right),$$

whence

$$N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, t\right) \geq N'\left(\frac{1}{\alpha^2}\varphi(x, x, x), \frac{t}{\sum_{k=m+1}^{n}(\frac{\alpha}{2})^{k+1}}\right), \quad (2.9)$$

for all $n > m \geq 0$, $x \in X$, and $t > 0$.

Fix $x \in X$. Since, $\lim_{s\to\infty} N'(\frac{1}{\alpha^2}\varphi(x, x, x), s) = 1$ and $\sum_{n=0}^{\infty}\left(\frac{\alpha}{2}\right)^n$ is convergent, $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $H(x) \in Y$. Define $H : X \to Y$ by

$$H(x) = N - \lim_{n\to\infty}\frac{f(2^n x)}{2^n},$$

for all $x \in X$. Let $m = 0$ in (2.9), we have

$$N\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq N'\left(\frac{1}{\alpha^2}\varphi(x, x, x), \frac{t}{\sum_{k=1}^{n}(\frac{\alpha}{2})^{k+1}}\right),$$

for all $x \in X$ and $t > 0$. By (N4), we have

$$N(H(x) - f(x), t) \geq \min\left\{N\left(H(x) - \frac{f(2^n x)}{2^n}, \frac{t}{2}\right), N\left(\frac{f(2^n x)}{2^n} - f(x), \frac{t}{2}\right)\right\}, \quad (2.10)$$

for all $x \in X$ and $t > 0$. Taking $n \to \infty$ in (2.10) and using (N6), we get

$$\begin{aligned} N(H(x) - f(x), t) &\geq \lim_{n\to\infty} N\left(\frac{f(2^n x)}{2^n} - f(x), \frac{t}{2}\right) \\ &\geq \lim_{n\to\infty} N'\left(\frac{1}{\alpha^2}\varphi(x, x, x), \frac{t}{2\sum_{k=1}^{n}(\frac{\alpha}{2})^{k+1}}\right) \end{aligned}$$

$$= N'\left(\frac{\varphi(x, x, x)}{2-\alpha}, t\right),$$

for all $x \in X$ and $t > 0$.

Next, we show H is $\mathbb{C}$-linear. By (N4),

$$N(C_\lambda H(x, y, z), t) \geq \min\left\{N\left(2H\left(\frac{\lambda x + \lambda y}{2} + \lambda z\right) - 2\frac{f\left(2^n\left(\frac{\lambda x+\lambda y}{2} + \lambda z\right)\right)}{2^n}, \frac{t}{5}\right),\right.$$

$$N\left(\lambda H(x) - \lambda\frac{f(2^n x)}{2^n}, \frac{t}{5}\right), N\left(\lambda H(y) - \lambda\frac{f(2^n y)}{2^n}, \frac{t}{5}\right),$$

$$\left.N\left(2\lambda H(z) - 2\lambda\frac{f(2^n z)}{2^n}, \frac{t}{5}\right), N\left(\frac{1}{2^n}C_\lambda f(2^n x, 2^n y, 2^n z), \frac{t}{5}\right)\right\}, \quad (2.11)$$

for all $x, y, z \in X$ and $t > 0$. Taking $n \to \infty$ in (2.11) and by (2.2) and (2.3), we have

$$N(C_\lambda H(x, y, z), t) \geq \lim_{n\to\infty} N\left(\frac{1}{2^n}C_\lambda f(2^n x, 2^n y, 2^n z), \frac{t}{5}\right)$$

$$\geq \lim_{n\to\infty} N'\left(\frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z), \frac{t}{5}\right) = 1,$$

for all $x, y, z \in X$ and $t > 0$. So, $C_\lambda H(x, y, z) = 0$ for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in X$. It follows from Lemma 2.2 of [14] that the mapping $H : X \to X$ is $\mathbb{C}$-linear.

It follows from (2.4) that

$$N(H([x, y, z]) - [H(x), H(y), H(z)], s)$$

$$\geq \min\left\{N\left(H([x, y, z]) - \frac{f([2^n x, 2^n y, 2^n z])}{2^{3n}}, \frac{s}{3}\right),\right.$$

$$N\left([H(x), H(y), H(z)] - \left[\frac{f(2^n x)}{2^n}, \frac{f(2^n y)}{2^n}, \frac{f(2^n z)}{2^n}\right], \frac{s}{3}\right),$$

$$N\left(2^{-3n}(f([2^n x, 2^n y, 2^n z]) - [f(2^n x), f(2^n y), f(2^n z)]), \frac{s}{3}\right)\Bigg\}, \quad (2.12)$$

for all $x, y, z \in X$ and $s > 0$. Taking $n \to \infty$ in (2.12) and by (2.2), we get

$$N(H([x, y, z]) - [H(x), H(y), H(z)], s)$$

$$\geq \lim_{n\to\infty} N\left(2^{-3n}(f([2^n x, 2^n y, 2^n z]) - [f(2^n x), f(2^n y), f(2^n z)]), \frac{s}{3}\right)$$

$$\geq \lim_{n\to\infty} N'\left(2^{-3n}\varphi(2^n x, 2^n y, 2^n z), \frac{s}{3}\right) = 1,$$

for all $x, y, z \in X$ and $s > 0$. Therefore

$$H([x, y, z]) = [H(x), H(y), H(z)],$$

for all $x, y, z \in X$.

Let $H' : X \to Y$ be another C^*-ternary homomorphism satisfying (2.5). Then

$$N(H(x) - H'(x), t) = \lim_{n\to\infty} N\left(\frac{1}{2^n}(f(2^n x) - H'(2^n x)), t\right)$$

$$\geq \lim_{n\to\infty} N'\left(\frac{1}{2^n}\frac{\varphi(2^n x, 2^n x, 2^n x)}{2-\alpha}, t\right) = 1,$$

for all $x \in X$ and $t > 0$. Thus, $H'(x) = H(x)$ and this proves the uniqueness of H. □

Corollary 2.2. *Let X be a normed linear space* (*with norm* $\|\cdot\|$) *and* (Y, N) *be a fuzzy Banach space. Denote by* N', *the fuzzy norm obtained as Example* 1.2 *on* $\mathbb{R}$. *Assume that* $\theta > 0$, $p < 1$, $0 < \alpha < 2$, *and* $2^p < \alpha$. *Suppose* $f : X \to Y$ *is an odd function such that*

$$N(C_\lambda f(x, y, z), t) \geq N'(\theta(\|x\|^p + \|y\|^p + \|z\|^p), t),$$

and

$$N(f([x, y, z]) - [f(x), f(y), f(z)], s) \geq N'(\theta(\|x\|^p + \|y\|^p + \|z\|^p), s),$$

for all $x, y, z \in X$ *and all* $t, s > 0$. *Then, there exists a unique* C^**-ternary homomorphism* $H : X \to Y$ *such that*

$$N(f(x) - H(x), t) \geq \frac{t(2-\alpha)}{t(2-\alpha) + 3\theta\|x\|^p},$$

for all $x \in X$ *and* $t > 0$.

Proof. Let

$$\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$

for all $x, y, z \in X$. Then

$$\begin{aligned} N'(\varphi(2x, 2y, 2z), t) &= \frac{t}{t + \theta(\|2x\|^p + \|2y\|^p + \|2z\|^p)} \\ &\geq \frac{t}{t + \alpha\theta(\|x\|^p + \|y\|^p + \|z\|^p)} \\ &= N'(\alpha\varphi(x, y, z), t) \end{aligned}$$

$$\lim_{n\to\infty} N'\left(\frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z), t\right) = \lim_{n\to\infty} \frac{t}{t + 2^{n(p-1)}\theta(\|x\|^p + \|y\|^p + \|z\|^p)} = 1,$$

for all $x, y, z \in X$ and all $t > 0$. It follows from Theorem 2.1 that there is a unique C^*-ternary homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq N'\left(\frac{\varphi(x, x, x)}{2-\alpha}, t\right) = \frac{t(2-\alpha)}{t(2-\alpha) + 3\theta\|x\|^p},$$

for all $x \in X$ and all $t > 0$. □

Theorem 2.3. *Let X be a linear space and* (Z, N') *be a fuzzy normed space. Let* $\varphi : X^3 \to Z$ *be a function such that for some* $\alpha \in (2, \infty)$

$$N'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), t\right) \geq N'(\varphi(x, y, z), \alpha t), \tag{2.13}$$

$$\lim_{n\to\infty} N'\left(2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), t\right) = 1, \tag{2.14}$$

for all $x, y, z \in X$ *and all* $t > 0$. *Let* (Y, N) *be a fuzzy Banach space and let* $f : X \to Y$ *be an odd function such that*

$$N(C_\lambda f(x, y, z), t) \geq N'(\varphi(x, y, z), t), \tag{2.15}$$

$$N(f([x, y, z]) - [f(x), f(y), f(z)], s) \geq N'(\varphi(x, y, z), s), \tag{2.16}$$

for all $x, y, z \in X$ *and all* $t, s > 0$. *Then, there exists a unique* C^*-*ternary homomorphism* $H : X \to Y$ *such that*

$$N(f(x) - H(x), t) \geq N'\left(\frac{\varphi(x, x, x)}{\alpha - 2}, t\right), \tag{2.17}$$

for all $x \in X$ *and* $t > 0$.

Proof. Put $\lambda = 1$ and $x = y = z$ in (2.15), we have

$$N(2f(2x) - 4f(x), t) \geq N'(\varphi(x, x, x), t), \tag{2.18}$$

for all $x \in X$ and all $t > 0$. Using (2.13) and induction on n, we can verify that

$$N'\left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), t\right) \geq N'(\varphi(x, x, x), \alpha^n t), \tag{2.19}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{2^n}$ in (2.18) and using (2.19), we get

$$N\left(2f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right), t\right) \geq N'(\varphi(x, x, x), \alpha^n t), \tag{2.20}$$

for all $x \in X$ and all $t > 0$. It follows that

$$N\left(2^{n-1} f\left(\frac{x}{2^{n-1}}\right) - 2^n f\left(\frac{x}{2^n}\right), 2^{n-2} t\right) \geq N'\left(\frac{1}{\alpha^2}\varphi(x, x, x), \alpha^{n-2} t\right),$$

and whence,

$$N\left(2^{n-1} f\left(\frac{x}{2^{n-1}}\right) - 2^n f\left(\frac{x}{2^n}\right), \left(\frac{2}{\alpha}\right)^{n-2} t\right) \geq N'\left(\frac{1}{\alpha^2}\varphi(x, x, x), t\right),$$

for all $n \geq 1$, $x \in X$, and $t > 0$. Then

$$N\left(2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right), \sum_{k=m+1}^{n}\left(\frac{2}{\alpha}\right)^{k-2} t\right)$$

$$= N\left(\sum_{k=m+1}^{n}\left(2^k f\left(\frac{x}{2^k}\right) - 2^{k-1} f\left(\frac{x}{2^{k-1}}\right)\right), \sum_{k=m+1}^{n}\left(\frac{2}{\alpha}\right)^{k-2} t\right)$$

$$\geq N'\left(\frac{1}{\alpha^2}\varphi(x, x, x), t\right),$$

and therefore,

$$N\left(2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) \geq N'\left(\frac{1}{\alpha^2}\varphi(x, x, x), \frac{t}{\sum_{k=m+1}^{n}\left(\frac{2}{\alpha}\right)^{k-2}}\right), \quad (2.21)$$

for all $n > m \geq 0$, $x \in X$, and $t > 0$.

Fix $x \in X$. Since, $\lim_{s \to \infty} N'\left(\frac{1}{\alpha^2}\varphi(x, x, x), s\right) = 1$ and that $\sum_{n=0}^{\infty}\left(\frac{2}{\alpha}\right)^n$ is convergent, $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $H(x) \in Y$. Hence, we can define $H : X \to Y$ by

$$H(x) = N - \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right),$$

for all $x \in A$. The rest of the proof is similar to the proof of Theorem 2.1.

□

Corollary 2.4. *Suppose X is a normed space with* $\|\cdot\|$ *and* (Y, N) *is a fuzzy Banach space. Denote by* N', *the fuzzy norm obtained as Example 1.2 on* $\mathbb{R}$. *Assume that* $\theta > 0$, $p > 1$, $\alpha \in (2, \infty)$, *and* $2^p < \alpha$. *Suppose* $f : X \to Y$ *is an odd function such that*

$$N(C_\lambda f(x, y, z), t) \geq N'(\theta(\|x\|^p + \|y\|^p + \|z\|^p), t),$$

and

$$N(f([x, y, z]) - [f(x), f(y), f(z)], s) \geq N'(\theta(\|x\|^p + \|y\|^p + \|z\|^p), s),$$

for all $x, y, z \in X$ *and all* $t, s > 0$. *Then, there exists a unique* C^*-*ternary homomorphism* $H : X \to Y$ *such that*

$$N(f(x) - H(x), t) \geq \frac{t(\alpha - 2)}{t(\alpha - 2) + 3\theta\|x\|^p},$$

for all $x \in X$ *and* $t > 0$.

Proof. Let

$$\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$

for all $x, y, z \in X$. Then

$$N'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), t\right) = \frac{t}{t + \theta(\| \frac{x}{2} \|^p + \| \frac{y}{2} \|^p + \| \frac{z}{2} \|^p)}$$

$$\geq \frac{\alpha t}{\alpha t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}$$

$$= N'(\varphi(x, y, z), \alpha t),$$

and

$$\lim_{n\to\infty} N'\left(2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right), t\right) = \lim_{n\to\infty} \frac{t}{t + 2^{n(1-p)}\theta(\|x\|^p + \|y\|^p + \|z\|^p)} = 1,$$

for all $x, y, z \in X$ and all $t > 0$. By Theorem 2.3, there is a unique C^*-ternary homomorphism $H : X \to Y$ such that

$$N(f(x) - H(x), t) \geq N'\left(\frac{\varphi(x, x, x)}{\alpha - 2}, t\right) = \frac{t(\alpha - 2)}{t(\alpha - 2) + 3\theta\|x\|^p},$$

for all $x \in X$ and all $t > 0$. □

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